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Chiral Lagrangian and spectral sum rules for dense two-color QCD

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Abstract: We analytically study two-color QCD with an even number of flavors at high baryon density. This theory is free from the fermion sign problem. Chiral symmetry is broken spontaneously by the diquark condensate. Based on the symmetry breaking pattern we construct the low-energy effective Lagrangian for the Nambu-Goldstone bosons. We identify a new epsilon-regime at high baryon density in which the quark mass dependence of the partition function can be determined exactly. We also derive Leutwyler-Smilga-type spectral sum rules for the complex eigenvalues of the Dirac operator in terms of the fermion gap. Our results can in principle be tested in lattice QCD simulations.

Keywords: Spontaneous Symmetry Breaking, Chiral Lagrangians, Sum Rules.

Contents

1. Introduction

Quantum chromodynamics (QCD) exhibits various phases under extreme conditions: the quark-gluon plasma phase at high temperature T and the color superconducting phase [1, 2] at low T and intermediate or large quark chemical potential μ. The quark-gluon plasma phase is being studied experimentally in ultrarelativistic heavy-ion collisions and theoretically in lattice QCD simulations. On the other hand, the physics of color superconductivity at intermediate μ, which is important for the interior of neutron stars, is not yet fully understood, although many possible phases have been proposed, e.g., the meson condensed phase, the crystalline Fulde-Ferrell-Larkin-Ovchinikov phase, the gluon condensed phase, etc. [2]. The case of asymptotically large μ is better understood because in this region rigorous weak-coupling calculations can be performed owing to asymptotic freedom, and the ground state has been shown to be the most symmetric, color-flavor-locked (CFL) phase [3].

Given the enormous theoretical and phenomenological interest in QCD at nonzero density, it is unfortunate that first-principle lattice simulations are extremely difficult because of the fermion sign problem: The Dirac determinant becomes complex for nonzero μ, and conventional Monte Carlo methods fail. However, there is a category of theories, including two-color QCD with fundamental fermions and QCD with an arbitrary number of colors and adjoint fermions, that are free from the sign problem because of an additional

anti-unitary symmetry. Although these theories are quite different from QCD in certain respects (e.g., the pattern of chiral symmetry breaking is different from that of real QCD), they also exhibit dynamical phenomena such as diquark condensation that are expected to occur in real QCD, and thus we hope that they can provide us with some clues about the phase structure of real QCD. Recent years have seen substantial progress in the understanding of these QCD-like theories at nonzero temperature and density, both analytically and numerically [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

In particular, two-color QCD has been studied intensively. Two-color QCD at small μ has been investigated by chiral perturbation theory in the mean-field approximation [6, 7], later generalized in various directions [9, 13, 14], and many intriguing phenomena have been revealed, such as the occurrence of a second-order phase transition at $\mu = m_\pi/2$, where m_π is the pion mass. Also, vector mesons have been included in the chiral Lagrangian, which led to the prediction of vector meson condensation [12, 28, 29]. However, these approaches are based on an expansion in small μ and cannot be used to probe the system at large μ.

On the other hand, at *asymptotically large* μ the formation of a BCS superfluid of diquark pairs is expected based on the existence of an attractive channel between quarks near the Fermi surface. The gap for quasiparticles has been calculated perturbatively in [30, 18]. In this case, chiral symmetry is spontaneously broken not by the chiral condensate but by the diquark condensate, and the pattern of chiral symmetry breaking is quite different from that at $\mu \sim 0$.

In this paper we study two-color QCD at large μ analytically. We construct the chiral Lagrangian for the Nambu-Goldstone (NG) bosons associated with the spontaneous breaking of chiral symmetry for the case of an even number of flavors and determine the form of the mass term by symmetries.[1] Furthermore we consider the system on a finite torus and identify the corresponding "ε-regime" (or "microscopic domain") of the theory in which the finite-volume physics is completely dominated by the zero-momentum modes of the NG bosons. In this regime we can calculate the finite-volume partition function analytically as a function of the light quark masses. From this result we derive a set of novel spectral sum rules for inverse powers of the Dirac eigenvalues. These sum rules are a generalization of the well-known Leutwyler-Smilga sum rules at $\mu = 0$ [35], but our analysis is more rigorous in the sense that the occurrence of spontaneous symmetry breaking is not assumed, but derived, from the microscopic dynamics of QCD based on asymptotic freedom. We also propose that a microscopic spectral density can be defined similarly to the $\mu = 0$ case, and that it should be a universal function depending only on the global symmetries of the system. We hope that our exact results could be an important stepping stone to a deeper understanding of two-color QCD (and maybe even real QCD) at nonzero μ.

Recently, spectral sum rules for inverse Dirac eigenvalues have been derived in the CFL phase of three-color QCD by two of us [36], but due to the sign problem it is not straightforward to test them directly on the lattice. On the other hand, the new sum rules obtained in this paper for two-color QCD can in principle be tested on the lattice. The

[1] See [31, 32, 33, 34] for a similar construction in the CFL phase of three-color QCD.

lattice spacing has to be sufficiently small compared to $1/\mu$ to avoid lattice artifacts, but this seems to be mainly a technical problem and not a fundamental one.

This paper is organized as follows. In section 2 we give an overview of the symmetry breaking patterns depending on the presence of the chiral condensate, the diquark condensate, and/or the chemical potential (see also the discussion in [7]). In section 3, after reviewing the hierarchy of scales near the Fermi surface, we construct the low-energy effective Lagrangian for the Nambu-Goldstone (NG) bosons at large μ based on the pattern of chiral symmetry breaking caused by the diquark condensate. We also derive a mass formula connecting the mass of the NG bosons to the quark mass and the fermion gap. In section 4 we define the new ε-regime and exactly determine the quark mass dependence of the partition function. We then derive spectral sum rules for the inverse eigenvalues of the Dirac operator at large μ. We conclude in section 5. In the appendix we give a detailed microscopic derivation of the mass terms appearing in the low-energy effective Lagrangian.

2. Symmetry breaking patterns

Note that in this paper we discuss the continuum theory. Symmetries and their breaking pattern on the lattice depend on the lattice regularization adopted for the fermions and must be analyzed separately [5, 8].

We first define our notation, which is based on the notation used in [7]. The fermionic part of the Lagrangian in Euclidean space reads $\bar{\psi}(\mathcal{D}(\mu)+\mathcal{M})\psi$ with the μ-dependent Dirac operator

$$\mathcal{D}(\mu) = \gamma_\nu D_\nu + \gamma_0 \mu \tag{2.1}$$

and the mass term

$$\mathcal{M} = \frac{1}{2}(1+\gamma_5)M + \frac{1}{2}(1-\gamma_5)M^\dagger\,. \tag{2.2}$$

Here, ψ is a short-hand notation for N_f flavors of two-color Dirac spinor fields transforming in the fundamental representation of $\mathrm{SU}(2)_{\text{color}}$. The γ_ν are hermitian γ-matrices, with $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$. The covariant derivative D_ν is an anti-hermitian operator so that the eigenvalues of $\gamma_\nu D_\nu$ are purely imaginary. M is the $N_f\times N_f$ quark mass matrix, and for real and degenerate quark masses the mass term simplifies to $\mathcal{M}=m\mathbf{1}_{N_f}$.

Because of the pseudo-reality of SU(2) we have $\mathcal{D}(\mu)\tau_2C\gamma_5=\tau_2C\gamma_5\mathcal{D}(\mu)^*$, where C is the charge conjugation operator and τ_2 is a generator of $\mathrm{SU}(2)_{\text{color}}$. Together with chiral symmetry, $\{\gamma_5,\mathcal{D}(\mu)\}=0$, this can be used to show that if λ is one of the eigenvalues of $\mathcal{D}(\mu)$, so are $-\lambda$, λ^*, and $-\lambda^*$. While this observation is sufficient to see that the fermion determinant $\det(\mathcal{D}(\mu)+\mathcal{M})$ is real, it does not necessarily imply that two-color QCD is free of the sign problem; the point is that λ and λ^* are degenerate if λ is real [8]. Consequently, a sign problem can arise in two-color QCD with an odd number of flavors. We also note that in the simultaneous presence of a quark chemical potential μ and an isospin chemical potential μ_I, a sign problem can arise in two-color QCD with an arbitrary number of flavors [9]. Therefore, in this paper we only consider an even number of flavors and $\mu_I=0$ so that the sign problem is absent.

For $M = 0$ and $\mu = 0$, the fermionic part of the Lagrangian is symmetric under $\mathrm{U}(2N_f)$. For $\mu \neq 0$, this symmetry is broken explicitly to $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$. In general, the chiral symmetry of the Lagrangian is not realized in the ground state of the theory. Below we list the symmetries realized in various phases of two-color QCD with N_f flavors of Dirac fermions in the fundamental representation [4, 6, 7].[2][3]

1. $\mu = 0$, $\langle\bar{\psi}\psi\rangle = 0$, $\langle\psi\psi\rangle = 0$: $\mathrm{SU}(2N_f)$ $[\supset (\mathbb{Z}_{2N_f})_A]$
2. $\mu = 0$, $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\psi\psi\rangle = 0$: $\mathrm{Sp}(2N_f)$ $[\supset (\mathbb{Z}_2)_A]$
3. $\mu \neq 0$, $\langle\bar{\psi}\psi\rangle = 0$, $\langle\psi\psi\rangle = 0$: $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B$ $[\supset (\mathbb{Z}_{2N_f})_A]$
4. $\mu \neq 0$, $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\psi\psi\rangle = 0$: $\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$ $[\supset (\mathbb{Z}_2)_A]$
5. $\mu \neq 0$, $\langle\bar{\psi}\psi\rangle = 0$, $\langle\psi\psi\rangle \neq 0$: $\mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$ $[\supset (\mathbb{Z}_2)_L \times (\mathbb{Z}_2)_R]$
6. $\mu \neq 0$, $\langle\bar{\psi}\psi\rangle \neq 0$, $\langle\psi\psi\rangle \neq 0$: $\mathrm{Sp}(N_f)_V$ $[\supset (\mathbb{Z}_2)_B]$

In the first two lines, $\mathrm{U}(1)_B$ is contained in $\mathrm{SU}(2N_f)$ and $\mathrm{Sp}(2N_f)$, respectively. $(\mathbb{Z}_{2N_f})_A$ is the anomaly-free subgroup of the axial $\mathrm{U}(1)_A$ symmetry. $(\mathbb{Z}_{2N_f})_A$ is already contained both in $(\mathbb{Z}_{N_f})_L \times \mathrm{U}(1)_B$ and in $(\mathbb{Z}_{N_f})_R \times \mathrm{U}(1)_B$. Our notation is such that $\mathrm{Sp}(2) \sim \mathrm{SU}(2)$, therefore $\mathrm{Sp}(n)$ is defined only if n is even. In the last two lines, an even number of flavors is assumed. The fifth line is the case relevant for this paper, and the corresponding symmetry breaking pattern will be discussed in more detail in section 3.

The color-singlet diquark field $\psi\psi$ is a short-hand notation for

$$\psi\psi \equiv \varepsilon_{ab}(\psi_a^T)^i C\gamma_5 I^{ij}\psi_b^j\,, \tag{2.3}$$

where C is the charge-conjugation matrix, a, b are color indices, and i, j are flavor indices, respectively. The $N_f \times N_f$ symplectic matrix I is defined as

$$I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{2.4}$$

where $\mathbf{1}$ is the $(N_f/2) \times (N_f/2)$ unit matrix. (For $N_f = 2$, I is equal to $-i\sigma_2$.) As long as s-wave condensation is assumed, there may in principle be not only a color- and flavor-antisymmetric condensate but also a color- and flavor-symmetric condensate. However, the single-gluon exchange interaction indicates that the antisymmetric condensate is energetically favored [22], and thus we will not consider the symmetric one in the following.

Besides the scalar diquark condensate $\psi^T C\gamma_5\psi$ in (2.3), there is in principle also a pseudo-scalar diquark condensate $\psi^T C\psi$. Although the two are not distinguished by a single-gluon exchange interaction, it has been shown [38, 39] that the instanton-induced interaction favors the former. Based on this observation we also neglect the latter condensate in the following.[4]

[2]There is no SU(2) global anomaly [37] since we deal with Dirac fermions.

[3]The symplectic group $\mathrm{Sp}(2n) = \{g \in U(2n) \mid g^T I g = I\}$, where I is defined in (2.4). The dimension of $\mathrm{Sp}(2n)$ is $2n^2 + n$, and its center is $\mathbb{Z}_2$ irrespective of n.

[4]In [6] it was rigorously shown by QCD inequalities, which hold even at nonzero chemical potential owing to the positivity of the Euclidean path-integral measure of two-color QCD, that for massless quarks the lightest meson is the 0^+ diquark, and thus a condensation (if it occurs) must occur in the channel $\psi^T C\gamma_5\psi$.

3. Effective theory for dense two-color QCD

We now proceed to construct the low-energy effective Lagrangian for two-color QCD at large μ. A similar construction for the CFL phase of three-color QCD has been performed in [31, 32, 33, 34].

3.1 Hierarchy of scales

We first give a summary of the energy spectrum near the Fermi surface. For simplicity, the discussion in this subsection will mainly be restricted to the chiral limit (i.e., $M = 0$). According to perturbative calculations at large μ [30, 40] we have $0 \simeq \langle\bar{\psi}\psi\rangle \ll \langle\psi\psi\rangle$.[5] Therefore chiral symmetry is broken by the *diquark condensate*, not by the chiral condensate. The symmetry breaking pattern is (see section 2)

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \to \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R\,. \tag{3.1}$$

From the seminal renormalization group analysis near the Fermi surface [30], the fermion gap Δ for $\mathrm{SU}(2)_{\mathrm{color}}$ is found to be [40]

$$\Delta \sim \mu g^{-5} \exp\left(-\frac{2\pi^2}{g}\right), \tag{3.2}$$

where $g \equiv g(\mu)$ is the small running coupling obtained from the one-loop beta-function as

$$g(\mu)^2 \simeq \frac{12\pi^2}{(11 - N_f)\ln(\mu/\Lambda_{\mathrm{SU}(2)})} \quad \text{for } \mu \gg \Lambda_{\mathrm{SU}(2)}\,. \tag{3.3}$$

We assume $N_f < 11$ to ensure asymptotic freedom. In particular, we have

$$\Lambda_{\mathrm{SU}(2)} \ll \Delta \ll \mu \tag{3.4}$$

for sufficiently large μ.

We first discuss the breaking of the $\mathrm{U}(1)_B$ and $\mathrm{U}(1)_A$ symmetries. The $\mathrm{U}(1)_B$ symmetry is spontaneously broken, with an associated NG boson H that is gapless irrespective of the current quark mass. The $\mathrm{U}(1)_A$ symmetry is broken explicitly by the anomaly/instantons for small μ, but this explicit symmetry breaking disappears as $\mu \to \infty$ because instantons are screened in this limit [41]. In the latter case the $\mathrm{U}(1)_A$ symmetry is broken spontaneously by the diquark condensate, and we have an associated NG boson η'. At $\mu < \infty$ the η' acquires a mass from the anomaly, but for $\mu \gg \Lambda_{\mathrm{SU}(2)}$ the contribution of the anomaly to $m_{\eta'}$ can be neglected, and the η' can be treated as a pseudo-NG boson [32]. In other words, at large μ it is a good approximation to treat $\mathrm{U}(1)_A$ as a non-anomalous symmetry and assume that the partition function is independent of the vacuum angle θ.

We now turn to the breaking of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ to $\mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$. The case of $N_f = 2$ is special since $\mathrm{SU}(2) \sim \mathrm{Sp}(2)$. In this case there is no additional NG boson.

[5] $\langle\bar{\psi}\psi\rangle \simeq 0$ was observed in lattice simulations even at intermediate μ [11].

For $N_f = 4$ and larger, the spectrum contains additional NG bosons π whose masses are lifted by a nonzero current quark mass. Their total number is

$$2(N_f^2 - 1) - 2(N_f^2/2 + N_f/2) = N_f^2 - N_f - 2\,. \tag{3.5}$$

Are there other light particles? Quarks acquire a gap Δ and decouple from the dynamics near the Fermi surface. What about SU(2) gluons? In two-color QCD, the diquark condensate (2.3) is a color singlet, so the SU(2) gluons do not acquire a mass through the Higgs mechanism. The absence below Δ of particles charged under SU(2) indicates that the medium is transparent for SU(2) gluons and that neither Debye screening nor Meissner effect occur [42]. For these reasons, the low-energy dynamics of gluons is simply described by the Lagrangian of SU(2) Yang-Mills theory. It has been found in [43] that the confinement scale of in-medium SU(2) gluodynamics is considerably diminished from the value $\sim \Lambda_{\mathrm{SU}(2)}$ at $\mu = 0$ to

$$\Lambda'_{\mathrm{SU}(2)} \sim \Delta \exp\left(-\frac{2\sqrt{2}\pi}{11}\frac{\mu}{g\Delta}\right) \ll \Lambda_{\mathrm{SU}(2)} \tag{3.6}$$

due to the polarization effect of the diquark condensate. Since $\mu/g\Delta$ is large, see (3.2), it follows that gluons become almost gapless at asymptotically large μ. A more quantitative analysis [18] even suggests that gluons are lighter than the η' for sufficiently large μ. However, since their coupling to NG bosons does not show up at leading order of the QCD weak-coupling calculations [34, 18], we assume in the following that gluons do not interact with NG bosons.

The discussion of this section is summarized in table 1. The explanation of why $m_\pi^2 = 0$ or $\propto m^2$ for large μ will be given in section 3.2.

	$\mu \ll \Lambda_{\mathrm{SU}(2)}$	$\mu \gg \Lambda_{\mathrm{SU}(2)}$
chiral symmetry	spontaneously broken	spontaneously broken
m_π^2	$\propto m$	0 or $\propto m^2$
instantons	abundant	suppressed
$\mathrm{U}(1)_B$ symmetry	intact	spontaneously broken
$\mathrm{U}(1)_A$ symmetry	anomalous	spontaneously broken
η'	heavy	light
condensate	$0 \simeq \langle\psi\psi\rangle \ll \lvert\langle\bar\psi\psi\rangle\rvert$	$0 \simeq \langle\bar\psi\psi\rangle \ll \langle\psi\psi\rangle$
mass gap (quarks)	$\sim \Lambda_{\mathrm{SU}(2)}$	$\sim \Delta \gg \Lambda_{\mathrm{SU}(2)}$
mass gap (gluons)	$\sim \Lambda_{\mathrm{SU}(2)}$	$\sim \Lambda'_{\mathrm{SU}(2)} \ll \Lambda_{\mathrm{SU}(2)}$

Table 1: Qualitative differences in the physics of two-color QCD at large and small μ. m denotes the (degenerate) current quark mass. N_f is assumed to be even. See text for further explanations. In the second column, $\langle\psi\psi\rangle \ll \lvert\langle\bar\psi\psi\rangle\rvert$ only if $\mu \ll m_\pi/2$.

3.2 Chiral Lagrangian

We now turn to the construction of the low-energy effective Lagrangian associated with the symmetry breaking pattern (3.1). We introduce dimensionless color-singlet $N_f \times N_f$

matrix fields D_L and D_R, where N_f is even and > 2 (the case $N_f = 2$ will be considered separately at the end of this subsection). In terms of the microscopic fields, they can be expressed as

$$(D_L)^{ij} \sim (\psi_L^T)^i C \psi_L^j\,, \qquad (D_R)^{ij} \sim (\psi_R^T)^i C \psi_R^j\,. \tag{3.7}$$

Under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_A \times \mathrm{U}(1)_B$ the quarks transform as

$$\psi_L \to \mathrm{e}^{i(\alpha+\beta)} g_L \psi_L\,, \qquad \psi_R \to \mathrm{e}^{-i(\alpha-\beta)} g_R \psi_R\,, \tag{3.8}$$

where $g_i \in \mathrm{SU}(N_f)_i$ $(i = L, R)$ and the phases α and β are associated with the $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ rotations, respectively. Thus

$$D_L \to g_L D_L g_L^T \mathrm{e}^{2i(\alpha+\beta)}\,, \qquad D_R \to g_R D_R g_R^T \mathrm{e}^{-2i(\alpha-\beta)}\,. \tag{3.9}$$

It is convenient to split the fields into U(1) parts A, V and the rest,

$$D_L \equiv \Sigma_L A^\dagger V\,, \qquad D_R \equiv \Sigma_R A V\,, \tag{3.10}$$

so that

$$\Sigma_i \to g_i \Sigma_i g_i^T \quad (i = L, R)\,, \qquad A \to A\mathrm{e}^{-2i\alpha}\,, \qquad V \to V\mathrm{e}^{2i\beta}\,. \tag{3.11}$$

The mass term $\bar\psi_L M \psi_R + \bar\psi_R M^\dagger \psi_L$ is invariant if we treat the mass matrix M as a spurion field and let

$$M \to g_L M g_R^\dagger \mathrm{e}^{2i\alpha}\,. \tag{3.12}$$

The effective Lagrangian composed of Σ_L, Σ_R, A, V, and M should be invariant under (3.11) and (3.12). There is no invariant combination that contains an odd number of factors of M, in contrast to [7] where the effective Lagrangian contained a term linear in M. This difference can be understood as follows. As N_f is even, we can take $g_L = -1$ and $g_R = 1$ (or $g_L = 1$ and $g_R = -1$), under which M transforms as $M \to -M$ while Σ_L, Σ_R, A, and V remain unchanged. Therefore M must appear in even powers. The point is that the chiral condensate $\langle\bar\psi_L\psi_R\rangle$ is negligible at large μ; otherwise a NG field $\tilde\Sigma \sim \bar\psi_L\psi_R$ would appear that transforms under $g_L = -1$ as $\tilde\Sigma \to -\tilde\Sigma$ so that odd powers of M could appear in the Lagrangian in combination with $\tilde\Sigma$. (Strictly speaking, the assumption that N_f is even is not essential here. We can reach the same conclusion from the fact that the explicit breaking of $\mathrm{U}(1)_A$ by instantons vanishes at large μ, as discussed in [32].)

At $O(M^2)$ the real-valued invariant combination is uniquely found to be

$$A^2\,\mathrm{Tr}(M\Sigma_R M^T \Sigma_L^\dagger) + \text{c.c.} \tag{3.13}$$

No mass term appears for V reflecting the fact that $\mathrm{U}(1)_B$ is not violated by a nonzero quark mass. Σ_L and Σ_R, which are decoupled from each other in the chiral limit, are now coupled by M. Replacing $\Sigma_{L,R}$ by $\Sigma_{L,R}^T$ does not yield new invariants. This is because the diquark condensate is formed in a flavor antisymmetric channel, i.e., $\Sigma_{L,R}^T = -\Sigma_{L,R}$ as already noted in section 2. In addition, we have $\Sigma\Sigma^\dagger = -\Sigma\Sigma^* = \mathbf{1}_{N_f}$ and $\det\Sigma = 1$ for $\Sigma \in \mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$, which also greatly reduces the number of nontrivial invariants.

We conclude that the effective Lagrangian in Minkowski space-time, to lowest order in the derivatives and the quark masses, and assuming $N_f > 2$ and even, is given by[6]

$$\begin{aligned}\mathcal{L} = \frac{f_H^2}{2}\Big\{|\partial_0 V|^2 - v_H^2|\partial_i V|^2\Big\} + \frac{N_f f_{\eta'}^2}{2}\Big\{|\partial_0 A|^2 - v_{\eta'}^2|\partial_i A|^2\Big\} \\ + \frac{f_\pi^2}{2}\,\mathrm{Tr}\,\Big\{|\partial_0\Sigma_L|^2 - v_\pi^2|\partial_i\Sigma_L|^2 + (L\leftrightarrow R)\Big\} - c\Delta^2\Big\{A^2\,\mathrm{Tr}(M\Sigma_R M^T\Sigma_L^\dagger) + \text{c.c.}\Big\}\,.\end{aligned} \tag{3.14}$$

Here, f_H, $f_{\eta'}$, and f_π are the decay constants of H, η', and π, respectively, and the v's are the corresponding velocities originating from the absence of Lorentz invariance in the medium. The coefficient c is calculable using the method of [34] and found to be

$$c = \frac{3}{4\pi^2}\,. \tag{3.15}$$

The detailed derivation of (3.15) is given in appendix A. Note that $\mathcal{L}$ has no dependence on the θ-parameter. The effective chemical potential of $O(M^2)$ (the so-called Bedaque-Schäfer term in the CFL phase of three-color QCD [44]) is not displayed here since it is suppressed by $\sim 1/\mu$ at large μ. Also, its contribution to the finite-volume partition function starts at $O(M^4)$, which is sufficiently small compared to the leading $O(M^2)$ term (see section 4).

From (3.14) we can derive a mass formula for π in the flavor-symmetric case $M = m\mathbf{1}_{N_f}$. For this purpose, we define the NG fields π_i^a ($i = L, R$, $a = 1, \ldots, (N_f^2 - N_f)/2 - 1$) corresponding to the coset space $\mathrm{SU}(N_f)_i/\mathrm{Sp}(N_f)_i$ as[7]

$$\Sigma_i = U_i I U_i^T\,, \qquad U_i = \exp\left(\frac{i\pi_i^a X^a}{2\sqrt{N_f}f_\pi}\right), \tag{3.16}$$

where I is defined in (2.4) and the X^a are the generators of the coset $\mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$ satisfying the normalization $\mathrm{Tr}(X^a X^b) = N_f\delta^{ab}$ and the relation $X^a I = I(X^a)^T$. Since the mass term in (3.14) mixes π_L and π_R, it is necessary to diagonalize the mass matrix for $\pi_{L,R}^a$ to obtain the genuine mass eigenvalues by setting

$$\Pi^a = \frac{1}{\sqrt{2}}(\pi_L^a + \pi_R^a)\,, \qquad \tilde{\Pi}^a = \frac{1}{\sqrt{2}}(\pi_L^a - \pi_R^a)\,. \tag{3.17}$$

The resulting mass formula for Π^a and $\tilde{\Pi}^a$ reads

$$m_{\Pi^a} = 0\,, \qquad f_\pi^2 m_{\tilde{\Pi}^a}^2 = 4c\Delta^2 m^2\,. \tag{3.18}$$

The massless modes Π^a correspond to simultaneous rotations of π_L^a and π_R^a in the same direction, while the massive modes $\tilde{\Pi}^a$ correspond to rotations in the opposite direction (a similar discussion can be found in [45]).[8] For the massive modes $\tilde{\Pi}^a$, (3.18) is of exactly the same form as the expression derived in [32], except for the numerical factor of c. However,

[6]The Wess-Zumino-Witten term is necessary for completeness of the theory, but it is irrelevant to the ensuing analysis and neglected in the following.

[7]For $N_f = 2$, $UIU^T = (\det U)I = I$ is constant, in accordance with the isomorphism $\mathrm{SU}(2) \sim \mathrm{Sp}(2)$.

[8]If the quark masses are nondegenerate, the Π^a modes acquire masses.

our "pions" for two colors are two-quark (qq) states, while those for three colors are four-quark ($\bar{q}\bar{q}qq$) states in [32].

Similarly, we can also derive a mass formula for the η' boson associated with the spontaneous breaking of $\mathrm{U}(1)_A$. We again assume the flavor-symmetric case $M = m\mathbf{1}_{N_f}$. The η' field is defined by

$$A = \exp\left(i \frac{\eta'}{\sqrt{N_f} f_{\eta'}} \right) . \tag{3.19}$$

Expanding to second order in terms of the η' field in (3.14) yields

$$f_{\eta'}^2 m_{\eta'}^2 = 4c\Delta^2 m^2 . \tag{3.20}$$

Since $f_{\pi,\eta'} \sim \mu$ [32] we obtain from (3.18) and (3.20) the relation

$$m_{\Pi,\eta'} \sim \frac{m\Delta}{\mu} . \tag{3.21}$$

Finally, we consider the simpler case $N_f = 2$. In this case, the "pions" disappear. The first line of (3.14) requires no change while the second line is replaced by

$$-c'\Delta^2 \Big\{ (\det M) A^2 + \text{c.c.} \Big\} \quad \text{with} \quad c' = \frac{3}{2\pi^2} . \tag{3.22}$$

The above value of c' corrects the value of $4/3\pi^2$ given in [18].[9]

4. Finite-volume analysis

4.1 Microscopic domain

In this subsection we identify the microscopic domain of dense two-color QCD and discuss some related intricacies. In the following, we neglect the H boson and the gluonic sector since they decouple from the dynamics of the NG bosons we are interested in and do not contribute to the quark mass dependence of the partition function.

Let us study two-color QCD on a Euclidean four-dimensional torus $\beta \times L \times L \times L$ with linear spatial extent L and near-zero temperature $\beta = 1/T \sim L$. As is well known, the chiral limit and the thermodynamic limit do not commute in general. A particularly interesting regime to consider is the "ε-regime" (or "microscopic domain") of the theory in which the zero-momentum modes of the NG bosons dominate the partition function [46]. In the present setting, this regime is defined by the inequalities

$$\frac{1}{\Delta} \ll L \ll \frac{1}{m_{\Pi,\tilde{\Pi},\eta'}} , \tag{4.1}$$

where $m_{\Pi,\tilde{\Pi},\eta'}$ is the mass of the NG bosons $(\Pi, \tilde{\Pi}, \eta')$ at large μ. The first inequality, $1/\Delta \ll L$, means that the contribution of heavy non-NG bosons to the partition function can be neglected. The second inequality means that if the box size is much smaller than the Compton

[9] We thank Thomas Schäfer for a communication on this point.

wave length of the NG bosons, the functional integral over NG boson fields reduces to the zero-mode integral over the coset space $\mathrm{SU}(N_f)_L/\mathrm{Sp}(N_f)_L \times \mathrm{SU}(N_f)_R/\mathrm{Sp}(N_f)_R \times \mathrm{U}(1)_A$.

We briefly comment on the meaning of symmetry breaking assumed here. It is well known that a global symmetry cannot be broken spontaneously in a finite volume.[10] However, in two-color QCD the diquark condensate is a color singlet, which enables us to study diquark condensation phenomena by adding a *gauge-invariant* external symmetry-breaking field. This method is commonly used in lattice simulations, which are always restricted to a finite volume. This situation is in stark contrast to the CFL phase of three-color QCD, where the four-quark state composed of left- and right-handed diquark condensates is a color singlet, but the diquark condensate itself is not [36].

4.2 Massless spectral sum rules

In this subsection we determine the dependence of the finite-volume partition function on the light quark masses and use it to derive a set of exact spectral sum rules for the eigenvalues of the Dirac operator $\mathcal{D}(\mu)$ defined in (2.1). Let us first consider $N_f \geq 4$ (the case $N_f = 2$ will be discussed at the end of this subsection). Since the symmetry breaking pattern $\mathrm{SU}(n) \to \mathrm{Sp}(n)$ is the same as at $\mu = 0$ we can directly borrow calculational tools from [49]. The finite-volume partition function in the microscopic domain (4.1) is given by

$$\begin{aligned} Z(M) &= \int\limits_{\mathrm{U}(1)_A} dA \int\limits_{\mathrm{SU}(N_f)_L/\mathrm{Sp}(N_f)_L} d\Sigma_L \int\limits_{\mathrm{SU}(N_f)_R/\mathrm{Sp}(N_f)_R} d\Sigma_R \; \exp\left[V_4 c\Delta^2\Big\{A^2\,\mathrm{Tr}(M\Sigma_R M^T\Sigma_L^\dagger) + \text{c.c.}\Big\}\right] \\ &= \int\limits_{\mathrm{U}(1)_A} dA \int\limits_{\mathrm{SU}(N_f)_L} dU_L \int\limits_{\mathrm{SU}(N_f)_R} dU_R \; \exp\left[-V_4 c\Delta^2\Big\{A^2\,\mathrm{Tr}(MU_R I U_R^T M^T U_L^* I U_L^\dagger) + \text{c.c.}\Big\}\right], \end{aligned} \tag{4.2}$$

where Z is normalized to 1 in the chiral limit (i.e., all group volumes are defined to be unity) and $V_4 \equiv L^3\beta$ is the Euclidean space-time volume. Note that Z has no θ-dependence. In going from the first to the second line of (4.2), we have used $\Sigma = UIU^T$ as in (3.16), but for computational convenience we have chosen $U \in \mathrm{SU}(N_f)$ rather than $U \in \mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$. The integral over U is independent of the additional degrees of freedom we have introduced because $UIU^T = I$ for $U \in \mathrm{Sp}(N_f)$.

From the invariance property of the Haar measure, (4.2) can be expanded as

$$Z(M) = 1 + c_2\,\mathrm{Tr}(M^\dagger M) + c_4^{(1)}(\mathrm{Tr}\,M^\dagger M)^2 + c_4^{(2)}\,\mathrm{Tr}(M^\dagger M)^2 + O\big(M^6\big)\,. \tag{4.3}$$

However, all terms of $O(M^{4k-2})$ ($k = 1, 2, \dots$) vanish because of the $\mathrm{U}(1)_A$ integral. In particular $c_2 = 0$, so the first nontrivial coefficients are $c_4^{(1)}$ and $c_4^{(2)}$, both proportional to $(V_4 c\Delta^2)^2$. To figure out their proportionality constants, we use the formulas given in [49, (4.11)-(4.17)],

$$\int\limits_{U(2N)} dU\;\exp\big\{\,\mathrm{Tr}(YUIU^T) + \text{c.c.}\big\} = 1 + \frac{2}{2N-1}\,\mathrm{Tr}(Y^\dagger Y) + O(Y^4) \tag{4.4}$$

[10] Although there are some exceptions such as large-N transitions [47] and dynamical supersymmetry breaking [48].

for any $2N \times 2N$ antisymmetric matrix Y. (I is given in (2.4) but here has dimension $2N$.) Comparing (4.2) and (4.4) suggests the substitutions $U \to AU_R$, $N \to N_f/2$, and $Y \to -V_4 c\Delta^2 M^T U_L^* I U_L^\dagger M$ (Y is antisymmetric). We find

$$\begin{aligned} Z(M) &= \int\limits_{\mathrm{SU}(N_f)_L} dU_L \left\{1 - (V_4 c\Delta^2)^2 \frac{2}{N_f - 1} \operatorname{Tr}(M^\dagger U_L I U_L^T M^* \cdot M^T U_L^* I U_L^\dagger M)\right\} + O(M^8) \\ &= 1 + (V_4 c\Delta^2)^2 \frac{2}{(N_f-1)^2} \left\{(\operatorname{Tr} M^\dagger M)^2 - \operatorname{Tr}(M^\dagger M)^2\right\} + O(M^8)\,. \end{aligned} \tag{4.5}$$

We note in passing that the above combination $(\operatorname{Tr} M^\dagger M)^2 - \operatorname{Tr}(M^\dagger M)^2$ also appeared in our previous work for the CFL phase [36]. The above expansion is to be matched later against the one derived directly from the fundamental QCD Lagrangian.

As a special case, let us consider the case of a flavor-symmetric mass term $M = m\mathbf{1}_{N_f}$ with real m. Now one of the two $\mathrm{SU}(N_f)$ integrals (which corresponds to the integral over the massless modes Π^a in (3.18)) trivially drops out, leaving the far simpler expression

$$Z(m) = \int\limits_{U(N_f)} dU \, \exp\Big[-V_4 c\Delta^2 m^2 \Big\{ \operatorname{Tr}(UIU^T I) + \text{c.c.}\Big\}\Big]\,. \tag{4.6}$$

A small consistency check can be done using (4.4) in (4.6) to obtain (4.5) with $M = m\mathbf{1}_{N_f}$. This expression is of the same form as the one at $\mu = 0$ [49, (4.11)], which can be rewritten as a Pfaffian [49, (5.13)]. By the same token (with the replacements $x \to 4c\Delta^2 V_4 m^2$, $\nu \to 0$, and $2N_f \to N_f$, respectively, in [49, (5.13)]) we readily obtain the simple expression[11]

$$Z(m) = \frac{1}{(N_f-1)!!} \operatorname{Pf}(A)\,, \tag{4.8}$$

where A is an $N_f \times N_f$ antisymmetric matrix with entries

$$A_{pq} \equiv (q-p) I_{p+q}(4c\Delta^2 V_4 m^2)\,, \qquad p, q = -\frac{N_f-1}{2}, \dots, \frac{N_f-3}{2}, \frac{N_f-1}{2} \tag{4.9}$$

and I_{p+q} denotes a modified Bessel function. The analytical calculation of (4.2) for non-degenerate quark masses (as has been done elegantly at $\mu = 0$ for Dyson index $\beta_D = 2$ [50, 51, 52, 53] as well as for $\beta_D = 1, 4$ [54]) is left for future work.

Having determined the light quark mass dependence of the partition function, we can now derive spectral sum rules for two-color QCD at large μ. The philosophy is well-known: By adding a weak external field to a given physical system and measuring its response thereto, one can gain information about properties of the system *in the absence* of the external field. In the present situation, the external field is a non-zero quark mass, and we can learn something about the properties of the Dirac spectrum *in the chiral limit.*

[11] The Pfaffian of an $n \times n$ antisymmetric matrix X (for even n) is defined as

$$\operatorname{Pf}(X) = \frac{1}{2^{n/2}(n/2)!} \sum_\sigma \operatorname{sgn}(\sigma) X_{\sigma(1)\sigma(2)} \dots X_{\sigma(n-1)\sigma(n)}\,, \tag{4.7}$$

where the sum runs over all permutations σ of $1, 2, \dots, n$. Note that this definition differs by a factor of $1/2^{n/2}$ from [49, (A.8)].

We start with the properly normalized partition function expressed in terms of the microscopic degrees of freedom,

$$Z(M) = \int [\mathcal{D}A] \det \left[\mathcal{D}(\mu) + \frac{1+\gamma_5}{2}M + \frac{1-\gamma_5}{2}M^\dagger\right] \mathrm{e}^{-S_g} \Big/ \int [\mathcal{D}A] \det{}^{N_f} \mathcal{D}(\mu)\, \mathrm{e}^{-S_g}\,, \quad (4.10)$$

where $S_g \equiv \int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu}/4$ is the gluonic action. Let us denote the complex eigenvalues of $\mathcal{D}(\mu)$ by $i\lambda_n$, where the λ_n are real for $\mu = 0$. In accordance with the discussion in section 3.1, we assume that at large μ there are no exact zero modes, i.e., all λ_n are nonzero. Since all eigenvalues come in pairs $(i\lambda_n, -i\lambda_n)$, we have

$$\begin{aligned} Z(M) &= \int [\mathcal{D}A] \prod_n{}' \det(\lambda_n^2 + M^\dagger M)\, \mathrm{e}^{-S_g} \Big/ \int [\mathcal{D}A] \Big(\prod_n{}' \lambda_n^2\Big)^{N_f} \mathrm{e}^{-S_g} \\ &= \left\langle \prod_n{}' \det\left(1 + \frac{M^\dagger M}{\lambda_n^2}\right)\right\rangle, \end{aligned} \quad (4.11)$$

where $\prod'_n$ (and later $\sum'_n$) denotes the product (sum) over all eigenvalues λ_n with $\mathrm{Re}\,\lambda_n > 0$, and

$$\langle \mathcal{O} \rangle \equiv \int [\mathcal{D}A]\, \mathcal{O} \Big(\prod_n{}' \lambda_n^2\Big)^{N_f} \mathrm{e}^{-S_g} \Big/ \int [\mathcal{D}A] \Big(\prod_n{}' \lambda_n^2\Big)^{N_f} \mathrm{e}^{-S_g}\,. \quad (4.12)$$

Because of the pseudo-reality of SU(2), the measure in (4.12) is real and positive definite (for even N_f). Using the relation

$$\det(1+\varepsilon) = 1 + \mathrm{Tr}\,\varepsilon + \frac{1}{2}[(\mathrm{Tr}\,\varepsilon)^2 - \mathrm{Tr}\,\varepsilon^2] + O(\varepsilon^3)\,, \quad (4.13)$$

$Z(M)$ in (4.11) can be expanded in terms of the quark mass matrix M as

$$\begin{aligned} Z(M) = 1 &+ \left\langle \sum_n{}' \frac{1}{\lambda_n^2} \right\rangle \mathrm{Tr}\, M^\dagger M + \left\langle \sum_{m<n}{}' \frac{1}{\lambda_m^2 \lambda_n^2} \right\rangle (\mathrm{Tr}\, M^\dagger M)^2 \\ &+ \frac{1}{2} \left\langle \sum_n{}' \frac{1}{\lambda_n^4} \right\rangle \Big\{ (\mathrm{Tr}\, M^\dagger M)^2 - \mathrm{Tr}(M^\dagger M)^2 \Big\} + O(M^6)\,. \end{aligned} \quad (4.14)$$

Comparing the formal expansion (4.14) with (4.5), we obtain novel spectral sum rules for two-color QCD at large μ,[12]

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^2} \right\rangle = \left\langle \sum_{m<n}{}' \frac{1}{\lambda_m^2 \lambda_n^2} \right\rangle = \left\langle \sum_n{}' \frac{1}{\lambda_n^6} \right\rangle = 0\,, \quad (4.15)$$

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^4} \right\rangle = (4c\Delta^2 V_4)^2 \frac{1}{4(N_f-1)^2}\,, \quad (4.16)$$

which can in principle be continued to arbitrary orders of M. Note again that we have assumed even N_f with $N_f \geq 4$. The sums appearing in (4.15) and (4.16) are in fact real-valued for any *fixed* gauge configuration, i.e., before averaging over gauge fields. This can be seen from the pseudo-reality of SU(2): If λ is one of the eigenvalues, so is λ^*.

[12] Sum rules for complex Dirac eigenvalues have been obtained earlier in a different context, i.e., at small μ in the conventional microscopic domain $m_\pi, \mu \ll 1/L \ll \Lambda_{\mathrm{QCD}}$, see, e.g., [55].

Let us now define the spectral density

$$\rho(\lambda) = \left\langle \sum_n \delta^2(\lambda - \lambda_n) \right\rangle , \tag{4.17}$$

which is positive definite since the path-integral measure is (and thus ρ allows for a natural probabilistic interpretation). Note that the gauge average in (4.17) involves massless quarks, see (4.12). The case of nonzero masses will be addressed in section 4.3.

Then (4.16) becomes

$$\frac{1}{(4c\Delta^2 V_4)^2} \int_{\mathbb{C}_+} d^2\lambda \, \frac{\rho(\lambda)}{\lambda^4} = \int_{\mathbb{C}_+} d^2 z \, \frac{\rho_s^{V_4}(z)}{z^4} = \frac{1}{4(N_f - 1)^2} \tag{4.18}$$

with $\mathbb{C}_+ \equiv \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ and

$$\rho_s^{V_4}(z) \equiv \frac{\pi^2}{3\Delta^2 V_4} \, \rho\left(\frac{\pi z}{\sqrt{3\Delta^2 V_4}} \right) , \tag{4.19}$$

where the numerical factor $4c = 3/\pi^2$ has been included in the definition to simplify some analytical results that will be derived in section 4.3. Equation (4.18) implies the existence of a microscopic spectral density defined by

$$\rho_s(z) \equiv \lim_{V_4 \to \infty} \rho_s^{V_4}(z). \tag{4.20}$$

Since the sum rule (4.18) is determined solely by the symmetry breaking pattern induced by the diquark condensate, we expect that (4.20) is a *universal* function fully determined by the global symmetries and independent of the details of the microscopic interactions. Our conjecture is corroborated by various facts known at $\mu = 0$ [56]:

- The Leutwyler-Smilga sum rules are satisfied to good accuracy by the Dirac eigenvalues computed in the instanton liquid model, a phenomenological model of QCD.

- The microscopic spectral correlation functions derived from chiral random matrix theory (χRMT) generate the Leutwyler-Smilga sum rules exactly. Furthermore, they coincide with those derived from the finite-volume partition function of QCD to leading order in the ε-regime, because χRMT and QCD have the same global symmetries.

- The microscopic spectral correlation functions predicted by χRMT have been verified in a number of lattice QCD simulations.

It is thus natural to expect that the way the thermodynamic limit of the spectral density near zero is approached is universal at large μ, too. Further insights would be obtained if we could construct the appropriate random matrix model, a topic we hope to address elsewhere.

It is interesting to ask how the vanishing sum rules (4.15) can be understood in terms of the symmetries of the distribution of the eigenvalues λ_n. One might conjecture that the eigenvalues occur not only in quartets $(\lambda, -\lambda, \lambda^*, -\lambda^*)$ but in octets $(\pm\lambda, \pm i\lambda, \pm\lambda^*, \pm i\lambda^*)$.

This would be a sufficient condition for all averages of the type λ^{-4k+2} ($k = 1, 2, \ldots$) to vanish. However, this idea is flawed, as can be understood from

$$\left\langle \left({\sum_n}' \frac{1}{\lambda_n^2} \right)^2 \right\rangle = (4c\Delta^2 V_4)^2 \frac{1}{4(N_f - 1)^2}\,, \tag{4.21}$$

which follows from (4.15) and (4.16). Thus the symmetries are not realized per configuration, but only after averaging. We therefore turn to the spectral density $\rho(\lambda)$. It satisfies $\rho(\lambda) = \rho(\lambda^*) = \rho(-\lambda) = \rho(-\lambda^*)$. We now show that (4.15) follows if $\rho(\lambda)$ also satisfies $\rho(\lambda) = \rho(i\lambda)$,

$$\int d^2\lambda \, \frac{\rho(\lambda)}{\lambda^{4k-2}} = \int d^2\lambda \, \frac{\rho(i\lambda)}{(i\lambda)^{4k-2}} = -\int d^2\lambda \, \frac{\rho(\lambda)}{\lambda^{4k-2}} = 0\,. \tag{4.22}$$

The assumption $\rho(\lambda) = \rho(i\lambda)$ does not contradict (4.16). We speculate that it is indeed a property of ρ. In view of so many symmetries, it is tempting to assume that $\rho(\lambda) = \rho(|\lambda|)$. Then

$$\int d^2\lambda \, \frac{\rho(\lambda)}{\lambda^4} = \int |\lambda| d|\lambda| \, \frac{\rho(|\lambda|)}{|\lambda|^4} \oint d\varphi \, \mathrm{e}^{-4i\varphi} = 0\,, \tag{4.23}$$

which contradicts (4.16). Thus $\rho(\lambda)$ is anisotropic.

We finally consider $N_f = 2$. According to (3.22), the microscopic partition function (4.2) is given by

$$\begin{aligned} Z(M) &= \int_{\mathrm{U}(1)_A} dA \, \exp\left\{ V_4 c' \Delta^2 \Big[(\det M) A^2 + \text{c.c.} \Big] \right\} \\ &= I_0(2V_4 c' \Delta^2 |\det M|) \\ &= 1 + \frac{1}{2}(V_4 c' \Delta^2)^2 \Big\{ (\mathrm{Tr}\, M^\dagger M)^2 - \mathrm{Tr}(M^\dagger M)^2 \Big\} + O(M^8)\,, \end{aligned} \tag{4.24}$$

where we used the Cayley identity for 2×2 matrices. Comparing (4.24) with (4.14) we are again led to the vanishing sum rules (4.15) and to

$$\left\langle {\sum_n}' \frac{1}{\lambda_n^4} \right\rangle = (c' \Delta^2 V_4)^2\,, \tag{4.25}$$

which again hints at the existence of the universal microscopic spectral density (4.20). Note that (4.25) is actually the same as (4.16) with $N_f = 2$ because $c' = 2c$. The latter equality follows from $\mathrm{Tr}(MIM^TI) = -2\det M$ for $N_f = 2$, see (A.5) and (A.6). In other words, the analysis of this subsection for $N_f \geq 4$ holds for $N_f = 2$ as well if we set $\Sigma_L = \Sigma_R = I$ (corresponding to the trivial coset space SU(2)/Sp(2)) and $U_L = U_R = \mathbf{1}$ in (4.2), and $U_L = \mathbf{1}$ in (4.5). Of course, the physics is different since there are no "pions" for $N_f = 2$.

At first sight, the above results are counter-intuitive in the following sense. Since the $i\lambda_n$ are the eigenvalues of the Dirac operator $\gamma_\nu D_\nu + \gamma_0\mu$ for asymptotically large μ, one naively expects $\lambda_n \sim \mu$, and the inverse moments of the eigenvalues should *decrease* for larger μ. But our exact results imply exactly the opposite![13] — What we have overlooked is the existence of the Fermi surface. The typical momentum is comparable to the Fermi momentum, hence $\gamma_\nu D_\nu$ is not necessarily small compared to $\gamma_0\mu$. Delicate cancellations between these two terms lead to the above nontrivial sum rules.

[13] Δ diverges as $\mu \to \infty$, see (3.2).

4.3 Massive spectral sum rules

It is known that the massless spectral sum rules at $\mu = 0$ due to Leutwyler and Smilga can be generalized to the massive case by an appropriate rescaling of masses [57], and that the double-microscopic spectral correlations (called so because masses and eigenvalues are rescaled simultaneously as the volume is taken to infinity) derived from random matrix theory reproduce these sum rules [58, 59, 60]. Here we show that the new spectral sum rules we derived at large μ can be generalized to the massive case in a similar fashion. Such an extension will be useful in future comparisons with lattice gauge simulations, where light dynamical fermions are computationally expensive.

We begin with the simplest case, i.e., $N_f = 2$ with masses m_1 and m_2. According to the result in the previous subsection, we have

$$\begin{aligned} Z &= I_0(\alpha m_1 m_2) \quad \text{with} \quad \alpha = 2c'\Delta^2 V_4 = \frac{3\Delta^2 V_4}{\pi^2} \\ &= \frac{1}{\mathcal{N}} \int [\mathcal{D}A] \prod_n{}' \left[(\lambda_n^2 + m_1^2)(\lambda_n^2 + m_2^2)\right] \mathrm{e}^{-S_g} , \end{aligned} \tag{4.26}$$

where $\mathcal{N}$ is a mass-independent normalization factor. Below we represent expectation values with respect to the massive measure by $\langle\!\langle \cdots \rangle\!\rangle$.

Differentiation of Z by the masses yields massive spectral sum rules, e.g.,[14]

$$\frac{\partial}{\partial(m_1^2)} \log Z = \left\langle\!\!\left\langle \sum_n{}' \frac{1}{\lambda_n^2 + m_1^2} \right\rangle\!\!\right\rangle = \frac{(\alpha m_2)^2}{4} \frac{I_0(\alpha m_1 m_2) - I_2(\alpha m_1 m_2)}{I_0(\alpha m_1 m_2)} , \tag{4.27}$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial(m_1^2)\partial(m_2^2)} = \left\langle\!\!\left\langle \sum_n{}' \frac{1}{\lambda_n^2 + m_1^2} \sum_k{}' \frac{1}{\lambda_k^2 + m_2^2} \right\rangle\!\!\right\rangle = \frac{\alpha^2}{4} , \tag{4.28}$$

$$\begin{aligned} \frac{1}{Z} \frac{\partial^2 Z}{\partial(m_1^2)^2} &= \left\langle\!\!\left\langle \sum_{n \neq k}{}' \frac{1}{(\lambda_n^2 + m_1^2)(\lambda_k^2 + m_1^2)} \right\rangle\!\!\right\rangle \\ &= \frac{(\alpha m_2)^4}{96} \frac{3I_0(\alpha m_1 m_2) - 4I_2(\alpha m_1 m_2) + I_4(\alpha m_1 m_2)}{I_0(\alpha m_1 m_2)} . \end{aligned} \tag{4.29}$$

For degenerate masses ($m_1 = m_2 \equiv m$) we have $Z = I_0(\alpha m^2)$ so that

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial(m^2)^2} = \left\langle\!\!\left\langle -2\sum_n{}' \frac{1}{(\lambda_n^2 + m^2)^2} + 4\left(\sum_n{}' \frac{1}{\lambda_n^2 + m^2} \right)^2 \right\rangle\!\!\right\rangle = \frac{\alpha^2}{2} \frac{I_0(\alpha m^2) + I_2(\alpha m^2)}{I_0(\alpha m^2)} . \tag{4.30}$$

In the massless limit equations (4.27)–(4.30) are consistent with the massless sum rules (4.15) and (4.25). In terms of rescaled dimensionless variables,

$$z_n \equiv \lambda_n \frac{\sqrt{3\Delta^2 V_4}}{\pi} , \qquad \tilde{m}_i \equiv m_i \frac{\sqrt{3\Delta^2 V_4}}{\pi} , \tag{4.31}$$

[14] We repeatedly used the identities $I_n'(x) = [I_{n-1}(x) + I_{n+1}(x)]/2$ and $I_n(x)/x = [I_{n-1}(x) - I_{n+1}(x)]/2n$.

equations (4.27)–(4.30) read

$$\left\langle\!\!\left\langle \sum_n{}' \frac{1}{z_n^2+\tilde{m}_1^2} \right\rangle\!\!\right\rangle = \frac{\tilde{m}_2^2}{4}\,\frac{I_0(x)-I_2(x)}{I_0(x)} \quad \text{with} \quad x=\tilde{m}_1\tilde{m}_2\,, \tag{4.32}$$

$$\left\langle\!\!\left\langle \sum_n{}' \frac{1}{z_n^2+\tilde{m}_1^2} \sum_k{}' \frac{1}{z_k^2+\tilde{m}_2^2} \right\rangle\!\!\right\rangle = \frac{1}{4}\,, \tag{4.33}$$

$$\left\langle\!\!\left\langle \sum_{n\neq k}{}' \frac{1}{(z_n^2+\tilde{m}_1^2)(z_k^2+\tilde{m}_1^2)} \right\rangle\!\!\right\rangle = \frac{\tilde{m}_2^4}{96}\,\frac{3I_0(x)-4I_2(x)+I_4(x)}{I_0(x)}\,, \tag{4.34}$$

$$\left\langle\!\!\left\langle -\sum_n{}' \frac{1}{(z_n^2+\tilde{m}^2)^2} + 2\Big(\sum_n{}' \frac{1}{z_n^2+\tilde{m}^2}\Big)^2 \right\rangle\!\!\right\rangle = \frac{I_0(\tilde{m}^2)+I_2(\tilde{m}^2)}{4I_0(\tilde{m}^2)}\,. \tag{4.35}$$

It is surprising that the right-hand side of (4.33) is independent of the masses.

We now define the double-microscopic spectral density by

$$\rho_s^{(N_f)}(z;\tilde{m}_1,\ldots,\tilde{m}_{N_f}) \equiv \lim_{V_4\to\infty} \frac{\pi^2}{3\Delta^2 V_4}\rho\left(\frac{\pi z}{\sqrt{3\Delta^2 V_4}}\right)\Bigg|_{\tilde{m}_i=m_i\frac{\sqrt{3\Delta^2 V_4}}{\pi}\ \text{fixed}}\,. \tag{4.36}$$

As mentioned before, the rescaling is the same for $N_f=2$ and $N_f\geq 4$ because $c'=2c$, and thus (4.36) applies to all (even) N_f. From the above sum rules we can derive several equalities characterizing $\rho_s^{(2)}$, e.g.,

$$\int_{\mathbb{C}_+} d^2z\,\frac{\rho_s^{(2)}(z;\tilde{m},\tilde{m})}{(z^2+\tilde{m}^2)^2} = \frac{I_0(\tilde{m}^2)-I_2(\tilde{m}^2)}{4I_0(\tilde{m}^2)}\,, \tag{4.37}$$

which is obtained from (4.35) and from (4.33) in the limit $\tilde{m}_1,\tilde{m}_2\to\tilde{m}$. The determination of $\rho_s^{(N_f)}$ itself requires further theoretical analysis and is left for future work.

For larger N_f the explicit expressions become increasingly involved. For $N_f=4$ with equal masses, we have

$$\left\langle\!\!\left\langle \sum_n{}' \frac{1}{z_n^2+\tilde{m}^2} \right\rangle\!\!\right\rangle = \frac{2I_0(y)I_1(y)-3I_1(y)I_2(y)+I_2(y)I_3(y)}{4Z} \qquad \text{with} \quad y=\tilde{m}^2\,, \tag{4.38}$$

where

$$Z = 3I_0(y)^2-4I_1(y)^2+3I_2(y)^2\,. \tag{4.39}$$

Analogously,

$$\left\langle\!\!\left\langle -\sum_n{}' \frac{1}{(z_n^2+\tilde{m}^2)^2} + 4\Big(\sum_n{}' \frac{1}{z_n^2+\tilde{m}^2}\Big)^2 \right\rangle\!\!\right\rangle = \frac{2I_0^2+I_1^2-2I_2^2+I_3^2-I_0I_2-2I_1I_3+I_2I_4}{8Z}\,. \tag{4.40}$$

One can check that the massless sum rules (4.15) and (4.16) are reproduced correctly in the chiral limit. Many more sum rules can be derived from the partition function for different fermion masses, but we do not work them out explicitly. We only mention that the sum rules corresponding to different masses satisfy nontrivial consistency conditions as some of the masses are sent to infinity, owing to decoupling [58].

5. Conclusions

In this paper we have constructed the low-energy effective Lagrangian for two-color QCD with an even number of flavors at large quark chemical potential μ based on the symmetry breaking pattern induced by the formation of a diquark condensate. For $N_f = 2$, there are two Nambu-Goldstone bosons H and η' associated with the spontaneous breaking of the $\mathrm{U}(1)_B$ and $\mathrm{U}(1)_A$ symmetries, respectively. The explicit breaking of $\mathrm{U}(1)_A$ vanishes at large μ due to the screening of instantons. For $N_f \geq 4$, there are additional NG bosons ("pions" π_L and π_R) associated with the chiral symmetry breaking pattern $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$. We derived a mass formula for the NG bosons in the case of degenerate quark masses. From the mixing of π_L and π_R by the mass term in the effective Lagrangian we found that half of them are exactly massless and that the other half are massive, with a mass proportional to the quark mass as a consequence of the $\mathbb{Z}(2)_L \times \mathbb{Z}(2)_R$ symmetry of the diquark pairing.

On the basis of the mass hierarchy near the Fermi surface, we have identified a new finite-volume ε-regime for the superfluid phase at large μ, where the mass scale of the non-NG modes is characterized by the fermion gap Δ. In this regime, we can exactly determine the quark mass dependence of the partition function from the effective theory. Matching this result against the two-color QCD partition function, we have derived novel spectral sum rules for inverse powers of the complex eigenvalues of the Dirac operator. Our sum rules explicitly show that the Dirac spectrum at large μ is governed by the fermion gap Δ, unlike the spectrum at low μ, which is governed by the chiral condensate as shown in [49].

A remarkable property of two-color QCD is that the fermion sign problem is absent at nonzero μ (for even N_f with pairwise degenerate quark masses) because of the pseudo-reality of SU(2). Therefore, our sum rules can in principle be checked in lattice QCD simulations. This is in contrast to real (three-color) QCD where the severe sign problem prevents us from observing the presumed color superconductivity, although similar spectral sum rules could be derived in the corresponding ε-regime [36]. Testing our sum rules for two colors on the lattice would enable us not only to verify the existence of the BCS superfluid phase originating from diquark pairing, but also to determine the value of the BCS gap Δ at large μ numerically. This is important since previous studies of two-color QCD at nonzero μ have only been able to determine the magnitude of the diquark condensate, not the gap itself.

One should note that the diquark condensate *does not necessarily* imply a nonzero BCS gap Δ. At lower μ, just above $m_\pi/2$, the diquark condensate is a strongly-coupled Bose-Einstein condensate (BEC) rather than the weakly-coupled BCS pairing expected at large μ [25], although there is the possibility of a crossover between these two states [9] (see, however, [25]). Also, the diquark condensate and the BCS gap are intrinsically different in the context of the Dirac spectrum in that the former (the latter) is governed by the radial (the anisotropic two-dimensional) distribution of Dirac eigenvalues [61]. We are planning to embark on a lattice simulation of two-color QCD at large μ to test the ideas proposed in this paper and to make a first step towards providing a direct signature of the BCS phase of two-color QCD (i.e., the gap around the Fermi surface) at large μ.

The BEC-BCS crossover, or the continuity between the hadronic phase and the BCS superfluid phase [62], may be closely related to the formal similarity of the partition functions for degenerate quark masses at small μ, see [49, (5.13)], and at large μ, see (4.8) obtained in this paper. In the case of real (three-color) QCD with three flavors, it was recently suggested that the axial anomaly (or the effect of instantons), which acts as an external field for the chiral condensate, can lead to a crossover between the hadronic phase and the color superconducting phase [63, 45, 64] as an explicit realization of the quark-hadron continuity in [62]. However, for QCD with two colors and an even numbers of flavors, the axial anomaly never acts as an external field for the chiral condensate, hence another mechanism to account for the crossover phenomenon would be necessary.

Finally, regarding the Dirac spectrum, it would be interesting to obtain an analogue of the Banks-Casher relation at large μ and the concrete form of the microscopic spectral density in the ε-regime we identified. In particular, it is a challenging problem to construct the corresponding random matrix model, which has turned out to be very successful at $\mu = 0$ and small μ, to reproduce our spectral sum rules and to elucidate universal properties of the Dirac spectrum at large μ for both two and three colors. (Earlier random matrix model approaches at large μ include [65, 66, 67, 10, 68, 21].) The partial quenching technique (see, e.g., [69, 70, 71]) may well be relevant in this regard. A detailed analysis of these issues is deferred to future work.

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A. Microscopic derivation of mass terms in the effective theory

In this appendix we give the detailed derivation of (3.15) and (3.22). For this purpose, we calculate the shift of the vacuum energy due to the quark mass from the microscopic theory (QCD), which should be matched against the result obtained from the low-energy effective Lagrangian (3.14). The starting point is the mass term of the microscopic theory given in [34],

$$\mathcal{L}_{\text{mass}} = \frac{g^2}{8\mu^4}(\psi^{a\dagger}_{i,L} C \psi^{b,\dagger}_{j,L})(\psi^c_{k,R} C \psi^d_{l,R}) \Big[(\tau^A)^{ac}(\tau^A)^{bd} M_{ik} M_{jl}\Big] + (L \leftrightarrow R,\, M \leftrightarrow M^\dagger)\,, \quad \text{(A.1)}$$

where i,j,k,l (a,b,c,d) are flavor (color) indices and τ^A $(A = 1,\ldots,N_c^2-1)$ are $\mathrm{SU}(N_c)$ color generators ($N_c = 2$ here) with normalization $\mathrm{Tr}(\tau^A\tau^B) = \delta^{AB}/2$. The diquark condensates d_L and d_R that are antisymmetric in flavor and color can be written as

$$\langle \psi^a_{i,L} C \psi^b_{j,L}\rangle = \varepsilon^{ab} I_{ij} d_L\,, \qquad \langle \psi^a_{i,R} C \psi^b_{j,R}\rangle = \varepsilon^{ab} I_{ij} d_R\,, \quad \text{(A.2)}$$

where I is defined in (2.4). The expectation values of d_L and d_R in the ground state can be evaluated in weak coupling. Generalizing the result of [34] to arbitrary N_c, we obtain

$$|d_L| = |d_R| = \sqrt{\frac{6N_c}{N_c+1}\frac{\mu^2\Delta}{\pi g}}\,. \tag{A.3}$$

Using (A.2), (A.3), and the Fierz identity

$$(\tau^A)^{ac}(\tau^A)^{bd} = \frac{1}{2}\delta^{ad}\delta^{bc} - \frac{1}{2N_c}\delta^{ac}\delta^{bd}\,, \tag{A.4}$$

the shift of the vacuum energy due to the quark mass is given by

$$\Delta\mathcal{E} = \frac{3\Delta^2}{4\pi^2}\,\mathrm{Tr}(MIM^TI) + (M \leftrightarrow M^\dagger)\,. \tag{A.5}$$

For $N_f = 2$ we have $I_{ij} = \varepsilon_{ij}$, and (A.5) reduces to

$$\Delta\mathcal{E}_{N_f=2} = -\frac{3\Delta^2}{2\pi^2}\det M + (M \leftrightarrow M^\dagger)\,. \tag{A.6}$$

By comparing (A.5) or (A.6) with the vacuum energy obtained from the low-energy effective Lagrangian (3.14) (with $\Sigma_L = \Sigma_R = I$) or (3.22), the values of the coefficients are found to be

$$c = \frac{3}{4\pi^2}\,, \qquad c' = \frac{3}{2\pi^2}\,, \tag{A.7}$$

respectively, where c' corrects the value of $4/3\pi^2$ given in [18].

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